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# Gas-Particulate Models of Flow through Porous Structures

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**ABSTRACT**
A recently developed general model of gas-particulate flow is sub-classified in this work. The model takes into account both the Darcy resistance and the Forchheimer effects, and is valid for variable particle number density and flow through variable porosity media. The form of governing equations is discussed when the particle relaxation time is small.
***Keywords*** - Saffman's model, Forchheimer and Darcy effects.

## I. INTRODUCTION

Modelling gas-particulate flow through porous media represents a fundamental step in the understanding of particle and carrier-fluid behaviours in the flow domain and their complex interactions with the solid matrix of the porous structure. [1]. [2], [3]. The diffusion and dispersion processes of the particle phase, particle reflection and capture mechanisms that take place in the porous structure, (cf. [4], [5], [6], [7]), and the interaction of the flowing phases, drives the efficient design of filtration systems, liquid-dust separators, and the transport of slurry through channels and pipes, [8], [9], [10]. In free-space, that is, in the absence of a porous matrix in the flow domain, a popular model that received considerable attention in the literature, is Saffman's dusty gas flow model [11] that assumes a small bulk concentration of dust particles per unit volume.

Success of implementation of Saffman's dusty gas model in theoretical and practical settings makes it an ideal candidate for describing dusty gas flow through porous structures. To this end, Saffman's equations are volume-averaged over a control volume, and the interactions of flowing phases with each other and their interactions with the porous structure are analyzed. Using this approach, a number of models have been developed and are reported in the literature, [12], [13], [14], [15], [16]. [17]. [18], and have ranges of validity from constant particle number density to variable number density, and porous structures ranging from isotropic granular to consolidated media, of either constant or variable porosity.

While in the vast majority of available models the effects of porous structure have been viewed in terms of Darcy resistance, a recent model. [18], has been developed to account for both the Darcy resistance and the Forchheimer effects that are important in capturing the effects of high-speed flow and the inertial effects that arise due to the converging-diverging pore structures and the tortuosity of the flow path. Due to the significance of particle inertia, it seems natural that Forchheimer effects are taken into account when modelling gas-particulate flow through porous media.

The model developed is one that describes the time-dependent flow of a dusty gas through an isotropic porous structure of variable porosity. However, the model is in disparate need for sub-classification in order to improve its practical utility in situations where assumptions are needed in the absence of experimental validation. This motivates the present work in which we start with the developed model and offer its sub-classifications. We also analyze the small relaxation time effect on the model equations.

## II. MODEL EQUATIONS

The time-dependent flow of a viscous, incompressible gas-particle mixture through an isotropic porous structure in which Darcy resistance and Forchheimer effects are taken into account, is governed by the following coupled set of intrinsically averaged equations, [18]:

*Fluid-phase averaged continuity equation*

$$\nabla \bullet \varphi < \vec{U} >_{\varphi} = 0. \qquad \ldots(1)$$

*Fluid-phase averaged momentum equation*

$$\rho_f \left[ \frac{\partial \varphi < \vec{U} >_{\varphi}}{\partial t} + \nabla \bullet \varphi < \vec{U} >_{\varphi} < \vec{U} >_{\varphi} \right] =$$

$$-\varphi\nabla < P >_{\varphi} + \mu\nabla^2\varphi < \vec{U} >_{\varphi} + K\varphi < N >_{\varphi} \left[< \vec{V} >_{\varphi} - < \vec{U} >_{\varphi}\right] - \rho_f \varphi < \vec{G} >_{\varphi}$$

$$-K\varphi\vec{\delta}[< N >_{\varphi} - N_d] - \frac{\mu}{\eta}\varphi < \vec{U} >_{\varphi} - \frac{\rho_f C_d}{\sqrt{\eta}}\varphi < \vec{U} >_{\varphi} \left|\varphi < \vec{U} >_{\varphi}\right|. \quad \ldots(2)$$

*Particle-phase averaged continuity equation*

$$\frac{\partial(\varphi < N >_{\varphi} - M)}{\partial t} + \nabla \bullet \varphi < N >_{\varphi} < \vec{V} >_{\varphi} + \nabla \bullet \varphi\vec{\delta}_2[< N >_{\varphi} - N_d] = 0 \quad \ldots(3)$$

*Particle-phase averaged momentum equation*

$$\rho_p\left[\frac{\partial}{\partial t}\varphi < N >_{\varphi} < \vec{V} >_{\varphi} + \nabla \bullet \varphi < N >_{\varphi} < \vec{V} >_{\varphi} < \vec{V} >_{\varphi}\right] = K\varphi < N >_{\varphi} (< \vec{U} >_{\varphi} - < \vec{V} >_{\varphi})$$

$$+ (\rho_f - \rho_p)\varphi < N >_{\varphi} < \vec{G} >_{\varphi} + K\varphi\vec{\delta}[< N >_{\varphi} - N_d] - \rho_p \frac{\partial}{\partial t}\varphi\vec{\delta}_2[< N >_{\varphi} - N_d]. \quad \ldots(4)$$

In equations (1)-(4), $\varphi$ is the medium porosity, $\eta$ is the permeability, $\mu$ is the fluid viscosity, $\rho_f$ is the fluid-phase mass density, $\rho_p$ is the particle-phase mass density, $K$ is the Stokes' coefficient of resistance, $C_d$ is the Forchheimer drag coefficient, $N_d$ is a reference average particle distribution, $\vec{\delta}, \vec{\delta}_1, \vec{\delta}_2$ are diffusion coefficient vectors, $< N >_{\varphi} - N_d$ is a number density driving differential, $\nabla$ is the gradient operator, and $\nabla^2$ is the laplacian. Other terms in the above equations are defined as follows. The intrinsic phase average of a quantity *F* (that is, the volumetric average of *F* over the effective pore space, $V_\phi$ ) is defined as

$$< F >_{\varphi} = \frac{1}{V_{\varphi}} \iiint_{V_\phi} F dV. \quad \ldots(5)$$

The averaged quantities are as follows: $< \vec{U} >_{\varphi}$ is the fluid-phase velocity, $< \vec{V} >_{\varphi}$ is the particle-phase velocity, $< N >_{\varphi}$ is the particle number density (particle distribution), $< \vec{G} >_{\varphi}$ is the body force (gravitational acceleration), $< P >_{\varphi}$ is the pressure, and $M$ is the total number of particles within a control volume, which can be approximated by:

$$M = \int_{V_{\varphi}} N dV \approx V_{\varphi} < N >_{\varphi}. \quad \ldots(6)$$

## III. CLASSIFICATION OF MODEL EQUATIONS

Equations (1)-(4) can be written in terms of specific discharge, defined for the fluid-phase and particle-phase, respectively, as:

$$\vec{q} \equiv \varphi < \vec{U} >_{\varphi} = \varphi\vec{u} \quad \ldots(7)$$

$$\vec{q}_d \equiv \varphi < \vec{V} >_{\varphi} = \varphi\vec{v}. \quad \ldots(8)$$

We employ the following notation: $< \vec{U} >_{\varphi} \equiv \vec{u}$; $< \vec{V} >_{\varphi} = \vec{v}$ $< N >_{\varphi} \equiv n$, $< \vec{G} >_{\varphi} \equiv \vec{g}$, and $< P >_{\varphi} = p$, and express equations (1)-(4) in the following forms, respectively:

*For fluid-phase:*

$$\nabla \bullet \vec{q} = 0. \quad \ldots(9)$$

$$\rho_f\left[\frac{\partial\vec{q}}{\partial t} + \nabla \bullet \frac{\vec{q}\vec{q}}{\varphi}\right] = -\varphi\nabla p + \mu\nabla^2\vec{q} + Kn[\vec{q}_d - \vec{q}] - \rho_f\varphi\vec{g} - K\varphi\vec{\delta}[n - N_d] \quad - \frac{\mu}{\eta}\vec{q} - \frac{\rho C_d}{\sqrt{\eta}}\vec{q}|\vec{q}| \quad \ldots(10)$$

*For dust-phase:*

$$\frac{\partial(\varphi n - M)}{\partial t} + \nabla \bullet n\vec{q}_d + \nabla \bullet \varphi\vec{\delta}_2[n - N_d] = 0 \qquad \text{...(11)}$$

$$\rho_p\left[\frac{\partial\{n\vec{q}_d + \varphi\vec{\delta}_2[n - N_d]\}}{\partial t} + \nabla \bullet \frac{n\vec{q}_d\vec{q}_d}{\varphi}\right] = \qquad \text{...(12)}$$

$$Kn(\vec{q} - \vec{q}_d) + (\rho_f - \rho_p)\varphi n\vec{g} + K\varphi\vec{\delta}[n - N_d]$$

### 3.1 THE CASE OF UNIFORM PARTICLE DISTRIBUTION

When the particle number density is constant (or the particle distribution is uniform), then its average is itself and its deviation is zero, so is its time rate of change. Equations (9)-(12) can them be written in the following forms, respectively:

*For fluid-phase:*

$$\nabla \bullet \vec{q} = 0. \qquad \text{...(13)}$$

$$\rho_f\left[\frac{\partial\vec{q}}{\partial t} + \nabla \bullet \frac{\vec{q}\vec{q}}{\varphi}\right] = -\varphi\nabla p + \mu\nabla^2\vec{q} + KN[\vec{q}_d - \vec{q}] - \rho_f\varphi\vec{g} - \frac{\mu}{\eta}\vec{q} - \frac{\rho C_d}{\sqrt{\eta}}\vec{q}|\vec{q}| \qquad \text{...(14)}$$

*For dust-phase:*

$$\nabla \bullet \vec{q}_d = 0 \qquad \text{...(15)}$$

$$\rho_p\left[\frac{\partial\vec{q}_d}{\partial t} + \nabla \bullet \frac{\vec{q}_d\vec{q}_d}{\varphi}\right] = K(\vec{q} - \vec{q}_d) + (\rho_f - \rho_p)\varphi\vec{g}. \qquad \text{...(16)}$$

### 3.2 STEADY STATE EQUATIONS

Equations (1)-(4) take the following steady-state form in terms of the intrinsic averaged phase velocities:

*For fluid-phase:*

$$\nabla \bullet \varphi\vec{u} = 0. \qquad \text{...(17)}$$

$$\rho_f\nabla \bullet \varphi\vec{u}\vec{u} = -\varphi\nabla p + \mu\nabla^2\varphi\vec{u} + K\varphi n[\vec{v} - \vec{u}] - K\varphi\vec{\delta}[n - N_d] - \frac{\mu}{\eta}\varphi\vec{u} - \frac{\rho C_d}{\sqrt{\eta}}\varphi\vec{u}|\varphi\vec{u}| \qquad \text{...(18)}$$

*For dust-phase:*

$$\nabla \bullet n\varphi\vec{v} + \nabla \bullet \varphi\vec{\delta}_2[n - N_d] = 0 \qquad \text{...(19)}$$

$$\rho_p\nabla \bullet n\varphi\vec{v}\vec{v} = Kn\varphi(\vec{u} - \vec{v}) + K\varphi\vec{\delta}[n - N_d]. \qquad \text{...(20)}$$

### 3.3 IGNORED DIFFUSION

When the particle distribution-driving differential is ignored due to ignoring diffusion and dispersion, equations (17)-(20) take the form:

*For fluid-phase:*

$$\nabla \bullet \varphi\vec{u} = 0. \qquad \text{...(21)}$$

$$\rho_f\nabla \bullet \varphi\vec{u}\vec{u} = -\varphi\nabla p + \mu\nabla^2\varphi\vec{u} + K\varphi n[\vec{v} - \vec{u}] - \frac{\mu}{\eta}\varphi\vec{u} - \frac{\rho C_d}{\sqrt{\eta}}\varphi\vec{u}|\varphi\vec{u}| \qquad \text{...(22)}$$

*For dust-phase:*

$$\nabla \bullet n\varphi\vec{v} = 0 \qquad \text{...(23)}$$

$$\rho_p\nabla \bullet n\varphi\vec{v}\vec{v} = Kn\varphi(\vec{u} - \vec{v}). \qquad \text{...(24)}$$

### 3.4 STEADY STATE EQUATIONS WITH CONSTANT POROSITY

If porosity is constant, equations (17)-(20) take the following forms, respectively:

*For fluid-phase:*

$$\nabla \bullet \vec{u} = 0. \quad \dots (25)$$

$$\rho_f \nabla \bullet \vec{u}\vec{u} = -\nabla p + \mu \nabla^2 \vec{u} + Kn[\vec{v} - \vec{u}] - K\vec{\delta}[n - N_d] - \frac{\mu}{\eta}\vec{u} - \frac{\rho C_d}{\sqrt{\eta}}\vec{u}|\varphi \vec{u}| \quad \dots (26)$$

*For dust-phase:*

$$\nabla \bullet n\vec{v} + \nabla \bullet \vec{\delta}_2[n - N_d] = 0 \quad \dots (27)$$

$$\rho_p \nabla \bullet n\vec{v}\vec{v} = Kn(\vec{u} - \vec{v}) + K\vec{\delta}[n - N_d]. \quad \dots (28)$$

### 3.5 CONSTANT POROSITY AND UNIFORM PARTICLE DISTRIBUTION

In the absence of particle number density deviations, equations (21)-(24) reduce to:
*For fluid-phase:*

$$\nabla \bullet \vec{u} = 0. \quad \dots (29)$$

$$\rho_f \nabla \bullet \vec{u}\vec{u} = -\nabla p + \mu \nabla^2 \vec{u} + Kn[\vec{v} - \vec{u}] - \frac{\mu}{\eta}\vec{u} - \frac{\rho C_d}{\sqrt{\eta}}\vec{u}|\varphi \vec{u}| \quad \dots (30)$$

*For dust-phase:*

$$\nabla \bullet \vec{v} = 0 \quad \dots (31)$$

$$\rho_p \nabla \bullet \vec{v}\vec{v} = K(\vec{u} - \vec{v}). \quad \dots (32)$$

### 3.6 EQUATIONS FOR SMALL RELAXATION TIME

Assuming that porosity is constant, the steady gas-particulate flow through an isotropic porous medium are given by equations (21)-(24). When porosity is constant, equation (24) can be written in the following form:

$$\frac{\rho_p}{K} \nabla \bullet n\vec{v}\vec{v} = n(\vec{u} - \vec{v}). \quad \dots (33)$$

Relaxation time of dust particles is the time required for a dust particle to adjust its path to that of the fluid elements. Relaxation time is thus defined by:

$$T = \frac{\rho_p}{K}. \quad \dots (34)$$

If $T$ is small (that is, $T \to 0$) then $\vec{v} \to \vec{u}$, as can be seen from equation (33). In this case, (23) becomes

$$\nabla \bullet n\vec{u} = 0 \quad \dots (35)$$

or

$$n + \vec{u} \bullet \nabla n = 0 \quad \dots (36)$$

With $\nabla \bullet \vec{u} = 0$, equation (36) becomes

$$\vec{u} \bullet \nabla n = 0 \quad \dots (37)$$

Equation (37) emphasizes that $n$ is constant on fluid-phase streamlines for two-dimensional flow. The fluid-phase momentum equations can then be written as:

$$\rho_f \nabla \bullet \vec{u}\vec{u} = -\nabla p + \mu \nabla^2 \vec{u} - \rho_p \nabla \bullet n\vec{v}\vec{v} - \frac{\mu}{\eta}\vec{u} - \frac{\rho C_d}{\sqrt{\eta}}\vec{u}|\varphi \vec{u}|. \quad \dots (38)$$

Using $\vec{v} = \vec{u}$ in (38) we obtain, after rearranging:

$$\rho_f \nabla \bullet \vec{u}\vec{u} + \rho_p \nabla \bullet n\vec{u}\vec{u} = -\nabla p + \mu \nabla^2 \vec{u} - \frac{\mu}{\eta}\vec{u} - \frac{\rho C_d}{\sqrt{\eta}}\vec{u}|\varphi \vec{u}|. \quad \dots (39)$$

Expanding the second term on LHS of (39), we obtain:

$$\rho_p \nabla \bullet n\vec{u}\vec{u} = \rho_p n \nabla \bullet \vec{u}\vec{u} + \vec{u}(\vec{u} \bullet \nabla n). \quad \dots (40)$$

Using (37) in (40) we obtain

$$\rho_p \nabla \bullet n\vec{u}\vec{u} = \rho_p n \nabla \bullet \vec{u}\vec{u}. \quad \dots (41)$$

Using (41), equation (39) becomes

$$(\rho_f + \rho_p n)\nabla \bullet \vec{u}\vec{u} = -\nabla p + \mu \nabla^2 \vec{u} - \frac{\mu}{\eta}\vec{u} - \frac{\rho C_d}{\sqrt{\eta}}\vec{u}\left|\varphi \vec{u}\right|. \quad \text{…(42)}$$

Equation (42) can be written in the following *non-dyadic*:

$$(\rho_f + \rho_p n)(\vec{u} \bullet \nabla)\vec{u} = -\nabla p + \mu \nabla^2 \vec{u} - \frac{\mu}{\eta}\vec{u} - \frac{\rho C_d}{\sqrt{\eta}}\vec{u}\left|\varphi \vec{u}\right|. \quad \text{…(43)}$$

The above analysis furnishes the establishment of the following Theorem:

**Theorem 1:** *The steady flow of a dusty gas with non-uniform particle distribution through an isotropic porous medium of constant porosity is governed by the following five scalar equations in the five unknowns* $\vec{u}$ *, n, and p*:

$$\nabla \bullet \vec{u} = 0. \quad \text{…(44)}$$

$$\vec{u} \bullet \nabla n = 0 \quad \text{…(45)}$$

$$(\rho_f + \rho_p n)(\vec{u} \bullet \nabla)\vec{u} = -\nabla p + \mu \nabla^2 \vec{u} - \frac{\mu}{\eta}\vec{u} - \frac{\rho C_d}{\sqrt{\eta}}\vec{u}\left|\varphi \vec{u}\right|. \quad \text{…(46)}$$

## IV. CONCLUSION

In this work, we considered a recently-developed general model that governs the unsteady flow of a gas-particle mixture with non-uniform particle distribution through an isotropic porous material. We provided a sub-classification of the model equations into forms suitable for the majority of flow situations, and provided a sub-model suitable for gas-particle flow with small relaxation time. This sub-classification may prove to be of assistance in finding approximate solutions to problems involving the general model.

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